Journal of Engineering Physics and Thermophysics. Vol. 73, No. 5, 2000

SURFACES OF DISCONTINUITY FOR A CUBICALLY ANISOTROPIC BODY IN THE MICROPOLAR THERMOELASTICITY THEORY

M. D. Martynenko and S. M. Bosyakov

UDC 539.3

We derived an equation for the propagation of thermoelastic waves in cubically anisotropic continuous media with account for the asymmetry of the stress tensor. By means of this equation the existence of longitudinal and transverse waves of displacements and microrotations is established.

We consider cubically anisotropic bodies for which the thermoelastic potential (with allowance for local rotations) has the form [1, 2]

$$\rho\Psi = \frac{1}{2}\left(A_1\varepsilon_{kk}^2 + 2A_2\varepsilon_{kk}\varepsilon_{ll} + A_3\varepsilon_{kl}^2 + 2A_4\varepsilon_{kl}\varepsilon_{lk}\right) +$$

$$+ \frac{1}{2}\left(B_1\varphi_{k,k}^2 + 2B_2\varphi_{k,k}\varphi_{l,l} + B_3\varphi_{k,l}^2 + 2B_4\varphi_{k,l}\varphi_{l,k}\right) - \beta\varepsilon_{kk}T \,. \quad (1)$$

Here $\varepsilon_{ji} = u_{ji} - \varepsilon_{ijm}\varphi_m$ is the tensor of microdeformations (micropolar deformations); $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$, $\vec{u} = (u_1, u_2, u_3)$, A_k, and B_k are the elastic and micropolar constants; β is the thermal-expansion coefficient; T is the absolute temperature; ε_{ijm} is the Levi-Civita tensor; $i, j, k, l, m, n = 1, 2, 3$.

From Eq. (1), for the tensors of force and moment stresses we obtain the following expressions:

$$t_{kk} = (A_1 - A_2)\,\varepsilon_{kk} + A_2 \sum_{i=1}^{3} \varepsilon_{ii} - \beta T \,, \quad t_{kl} = A_3\varepsilon_{kl} + A_4\varepsilon_{lk} \,,$$

$$m_{kk} = (B_1 - B_2)\,\varphi_{k,k} + B_2 \sum_{i=1}^{3} \varphi_{i,i} \,, \quad m_{kl} = B_4\varphi_{k,l} + B_3\varphi_{l,k} \,, \quad (2)$$

$$k \neq l = \overline{1, 3} \,.$$

Substitution of Eq. (2) into the equations of motion [2] gives

$$t_{lk,l} + \rho\left(f_k - \frac{\partial^2 u_k}{\partial t^2}\right) = 0 \,,$$

$$m_{lk,l} + \varepsilon_{kmn}t_{mn} + \rho\left(l_k - j\frac{\partial^2 \varphi_k}{\partial t^2}\right) = 0 \,.$$

As a result, we come to such a resolving system of equations for the components of the displacement and microrotation vectors as

Belarusian State University, Minsk. Translated from Inzhenerno-Fizcheskii Zhurnal, Vol.73, No. 5, pp. 1027-1032, September–October, 2000. Original article submitted March 16, 2000.

1062-0125/00/7305-1004$25.00

$$A_3 u_{k,ll} + A u_{k,kk} + (A_2 + A_4)\, u_{l,lk} + (A_3 - A_4)\, \varepsilon_{klm} \varphi_{m,l} + \rho f_k = \rho \frac{\partial^2 u_k}{\partial t^2} + \beta T_{,k} ,$$

$$B_3 \varphi_{k,ll} + B \varphi_{k,kk} + (B_2 + B_4)\, \varphi_{l,lk} + (A_3 - A_4)\, \varepsilon_{klm} u_{m,l} -$$

$$- 2\,(A_3 - A_4)\, \varphi_k + \rho l_k = j\rho \frac{\partial^2 \varphi_k}{\partial t^2} . \tag{3}$$

In Eqs. (3) $A = A_1 - A_2 - A_3 - A_4$ and $B = B_1 - B_2 - B_3 - B_4$.

The cubically anisotropic bodies behave with respect to their thermal expansion as isotropic bodies, i.e., they have one thermal-expansion coefficient β and one thermal-conductivity coefficient λ [3-5]. Therefore, to describe thermal processes in these bodies, we use a hyperbolic heat-conduction law for a disconnected body in the form [4, 5] (internal heat sources are absent):

$$\lambda \Delta T = c_\varepsilon \left(\frac{\partial T}{\partial t} + \tau \frac{\partial^2 T}{\partial t^2} \right), \tag{4}$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$.

Then, with allowance for the temperature effects, system (3) takes the form

$$A_3 u_{k,ll} + A u_{k,kk} + (A_2 + A_4)\, u_{l,lk} + (A_3 - A_4)\, \varepsilon_{klm} \varphi_{m,l} + \rho f_k = \rho \frac{\partial^2 u_k}{\partial t^2} + \beta T_{,k} ,$$

$$B_3 \varphi_{k,ll} + B \varphi_{k,kk} + (B_2 + B_4)\, \varphi_{l,lk} + (A_3 - A_4)\, \varepsilon_{klm} u_{m,l} -$$

$$- 2\,(A_3 - A_4)\, \varphi_k + \rho l_k = j\rho \frac{\partial^2 \varphi_k}{\partial t^2} . \tag{5}$$

$$\lambda \Delta T = c_\varepsilon \left(\frac{\partial T}{\partial t} + \tau \frac{\partial^2 T}{\partial t^2} \right).$$

Let us assume that the first derivatives of the components of the displacement and microrotation vectors and absolute temperature have strong discontinuities on a smooth surface $Z(t, x_1, x_2, x_3) = 0$. In this case the derivatives $\frac{\partial u_k}{\partial x_i}, \frac{\partial u_k}{\partial t}, \frac{\partial \varphi_k}{\partial x_i}, \frac{\partial \varphi_k}{\partial t}, \frac{\partial T}{\partial x_k}, \frac{\partial T}{\partial t}$, $i, k = \overline{1, 3}$, are continuous on each side of the surface $Z = 0$. Then, given the continuity of the functions u, φ, and T in passage through $Z(v, X)$ the following expressions remain continuous [6, 7]:

$$p_k \frac{\partial u_i}{\partial t} - p_0 \frac{\partial u_i}{\partial x_k} = M_{ki} , \tag{6}$$

$$p_k \frac{\partial \varphi_i}{\partial t} - p_0 \frac{\partial \varphi_i}{\partial x_k} = M_{k+3i+3} , \tag{7}$$

$$p_k \frac{\partial T}{\partial t} - p_0 \frac{\partial T}{\partial x_k} = M_k , \tag{8}$$

where M_{ki} are certain continuous functions, $p_0 = \partial Z/\partial t$, and $p_k = \partial Z/\partial x_k$, $i, k = \overline{1, 3}$.

Apart from the kinematic conditions (6)-(8), in passage through the surface of discontinuity the dynamic consistency conditions must be satisfied [6, 7]:

$$\sum_{k=1}^{3} t_{ik} p_k - \rho p_0 \frac{\partial u_i}{\partial t} = M_{4i}\,, \tag{9}$$

$$\sum_{k=1}^{3} m_{ik} p_k - \rho j p_0 \frac{\partial \varphi_i}{\partial t} = M_{4i+3}\,, \tag{10}$$

$$\lambda \sum_{k=1}^{3} \frac{\partial T}{\partial x_k} p_k - c_\varepsilon \tau p_0 \frac{\partial T}{\partial t} = M_7\,, \quad i = \overline{1, 3}\,. \tag{11}$$

From the system of equations (6)-(11) we can determine all the derivatives of the first order of u_i, φ_i, T, $i = \overline{1, 3}$. In order to simplify the calculation, we reduce system (6)-(11) to a simpler form. To do this, we multiply Eqs. (9)-(11) by p_0 and replace the expressions $p_0 \dfrac{\partial u_m}{\partial x_n}$, $p_0 \dfrac{\partial \varphi_m}{\partial x_n}$, and $p_0 \dfrac{\partial T}{\partial x_n}$, $m, n = \overline{1, 3}$, by the left sides of the equalities

$$p_k \frac{\partial u_i}{\partial t} - M_{ki} = p_0 \frac{\partial u_i}{\partial x_k}\,,$$

$$p_k \frac{\partial \varphi_i}{\partial t} - M_{k+3i+3} = p_0 \frac{\partial \varphi_i}{\partial x_k}\,,$$

$$p_k \frac{\partial T}{\partial t} - M_k = p_0 \frac{\partial T}{\partial x_k}\,, \quad i, k = \overline{1, 3}\,.$$

This leads to the following system of seven equations for the first derivatives of u_i, φ_i, and T with respect to t:

$$M_{4i}\, p_0 = \frac{\partial u_i}{\partial t}\left(A_3 g^2 + A p_i^2 - \rho p_0^2\right) + (A_2 + A_4)\, p_i \left(\frac{\partial u_j}{\partial t} p_j + \frac{\partial u_k}{\partial t} p_k\right) + \ldots\,,$$

$$M_{4i+3}\, p_0 = \frac{\partial \varphi_i}{\partial t}\left(B_3 g^2 + B p_i^2 - j\rho p_0^2\right) + (B_2 + B_4)\, p_i \left(\frac{\partial \varphi_j}{\partial t} p_j + \frac{\partial \varphi_k}{\partial t} p_k\right) + \ldots\,,$$

....

$$M_7 p_0 = \frac{\partial T}{\partial t}\left(\lambda g^2 - c_\varepsilon \tau p_0^2\right) + \ldots\,, \quad j, i, k = \overline{1, 3}\,. \tag{12}$$

The unsolvability of system (12) provides the condition for the fact that the partial derivatives of the first order $\dfrac{\partial u_k}{\partial x_i}$, $\dfrac{\partial u_k}{\partial t}$, $\dfrac{\partial \varphi_k}{\partial x_k}$, $\dfrac{\partial \varphi_k}{\partial t}$, $\dfrac{\partial T}{\partial x_k}$, and $\dfrac{\partial T}{\partial t}$, $i, k = \overline{1, 3}$, can have discontinuities on the surface $Z(t, X) = 0$. This means that in order to find the equation for the surface of discontinuity, we equate to zero the principal determinant of system (12): det $\Omega = 0$, where Ω is the matrix composed of the coefficients of these derivatives. Finally we obtain

$$((A_3 g^2 - \rho p_0^2)^3 + g^2 (A + A_2 + A_4)(A_3 g^2 - \rho p_0^2)^2 + A(A + 2(A_2 + A_4)) \times$$

$$\times (A_3 g^2 - \rho p_0^2)(p_1^2 p_2^2 + p_2^2 p_3^2 + p_1^2 p_3^2) + A^2 (A + 3(A_2 + A_4)) p_1^2 p_2^2 p_3^2) \times$$

$$\times ((B_3 g^2 - j\rho p_0^2)^3 + g^2 (B + B_2 + B_4)(B_3 g^2 - j\rho p_0^2)^2 + B(B + 2(B_2 + B_4)) \times$$

$$\times (B_3 g^2 - j\rho p_0^2)(p_1^2 p_2^2 + p_2^2 p_3^2 + p_1^2 p_3^2) + B^2 (B + 3(B_2 + B_4)) p_1^2 p_2^2 p_3^2) \times$$

$$\times (\lambda g^2 - c_\varepsilon \tau p_0^2) = 0 . \qquad (13)$$

Here $g^2 = p_1^2 + p_2^2 + p_3^2$.

It should be noted that Eq. (13) of the strong discontinuities of the displacement, microrotation, and temperature fields coincides with the equation of characteristics for system (5). The characteristic equation of system (5) can be obtained by assigning the initial data on the surface $Z = Z(t, x_1, x_2, x_3)$. For this purpose, we pass to new variables from the formulas

$$t = Z(t, X), \quad x_k = Z_k(t, X), \quad k = \overline{1, 3} . \qquad (14)$$

Then

$$\frac{\partial y(t, X)}{\partial x_k} = \sum_{i=0}^{3} \frac{\partial y}{\partial Z_i} \frac{\partial Z_i}{\partial x_k},$$

$$\frac{\partial^2 y}{\partial x_k \partial x_n} = \sum_{i,j=0}^{n} \frac{\partial^2 y}{\partial Z_j \partial Z_i} \frac{\partial Z_i}{\partial x_k} \frac{\partial Z_j}{\partial x_n} + \sum_{i=0}^{3} \frac{\partial y}{\partial Z_i} \frac{\partial^2 Z_i}{\partial x_n \partial x_k},$$

$$Z \equiv Z_0, \quad t \equiv x_0 . \qquad (15)$$

Now we substitute formulas (15) into the equations of system (5) and write those terms that contain $\frac{\partial^2 u_k}{\partial Z^2}$, $\frac{\partial^2 \varphi_k}{\partial Z^2}$, $\frac{\partial^2 T}{\partial Z^2}$, $k = \overline{1, 3}$:

$$(A_3 g^2 + A p_k^2 - \rho p_0^2) \frac{\partial^2 u_k}{\partial Z^2} + (A_2 + A_4) p_k \sum_{i=1}^{3} p_i \frac{\partial^2 u_i}{\partial Z^2} + \ldots = 0 ,$$

$$(B_3 g^2 + B p_k^2 - j\rho p_0^2) \frac{\partial^2 \varphi_k}{\partial Z^2} + (B_2 + B_4) p_k \sum_{i=1}^{3} p_i \frac{\partial^2 \varphi_i}{\partial Z^2} + \ldots = 0 , \qquad (16)$$

$$\frac{\partial^2 T}{\partial Z^2} (\lambda g^2 - c_\varepsilon \tau p_0^2) + \ldots = 0 .$$

The surface $Z = 0$ is characteristic for system (16) [6, 7], provided that simultaneously with the initial data it does not allow one to determine the second-order time derivatives. This is equivalent to the zero equality of the principal determinant of the system

$$\det \begin{Vmatrix} \Omega_1 & \Omega_2 & 0 \\ \Omega_3 & \Omega_4 & 0 \\ 0 & 0 & \Omega_5 \end{Vmatrix} = 0, \tag{17}$$

where

$$\Omega_1 = \|a_{ij}\|, \quad \Omega_2 = \Omega_3 = 0, \quad \Omega_4 = \|a_{i+3,j+3}\|, \quad i, j = \overline{1, 3};$$

$$a_{ii} = A_3 g^2 + (A_1 - A_3) p_i^2 - \rho p_0^2;$$

$$a_{ij} = a_{ji} = (A_2 + A_4) p_i p_j;$$

$$a_{i+3,i+3} = B_3 g^2 + (B_1 - B_3) p_i^2 - j\rho p_0^2;$$

$$a_{i+3,j+3} = a_{j+3,i+3} = (B_2 + B_4) p_i p_j;$$

$$\Omega_5 = \lambda g^2 - c_\varepsilon \tau p_0^2, \quad i, j = \overline{1, 3}.$$

Expanding the determinant, we come to Eq. (13).

Thus, the fields of the displacement $\vec{u}$, microrotation $\vec{\varphi}$, and temperature T with strong discontinuities in the first-order partial derivatives at the points of the surface $Z(t, X) = 0$ exist in the case where the surface $Z(t, X) = 0$ turns out to be characteristic for system (5).

Equation (13) allows the conclusion that in cubically anisotropic media there are seven types of surfaces of strong discontinuities that propagate at the velocities $v = -p_0/g$ [5, 6]. We use the following notation: v_1 and v_4 for the velocity of the longitudinal waves of displacement and microrotation; (v_2, v_3) (v_5, v_6) for the velocity of the transverse waves of displacement and microrotation; v for the velocity of the quasiheat wave. The equation for determining the velocities of these types of waves can be obtained by dividing both parts of Eq. (13) into g^{14}. As a result, we will have

$$\begin{aligned} &((A_3 - \rho v^2)^3 + (A + A_2 + A_4)(A_3 - \rho v^2)^2 + A(A + 2(A_2 + A_4)) \times \\ &\times (A_3 - \rho v^2)(\cos^2\alpha_1 \cos^2\alpha_2 + \cos^2\alpha_2 \cos^2\alpha_3 + \cos^2\alpha_1 \cos^2\alpha_3) + \\ &+ A^2 (A + 3(A_2 + A_4)) \cos^2\alpha_1 \cos^2\alpha_2 \cos^2\alpha_3) \times \\ &\times ((B_3 - j\rho v^2)^3 + (B + B_2 + B_4)(B_3 - j\rho v^2)^2 + B(B + 2(B_2 + B_4)) \times \\ &\times (B_3 - j\rho v^2)(\cos^2\alpha_1 \cos^2\alpha_2 + \cos^2\alpha_2 \cos^2\alpha_3 + \cos^2\alpha_1 \cos^2\alpha_3) + \\ &+ B^2 (B + 3(B_2 + B_4)) \cos^2\alpha_1 \cos^2\alpha_2 \cos^2\alpha_3)(\lambda g^2 - c_\varepsilon \tau v^2) = 0. \end{aligned} \tag{18}$$

Assigning the direction of the normal $\vec{n}$ to the surface element dZ at the point N, from Eq. (18) it is possible to calculate the velocities of propagation of the indicated types of waves. This can be done most simply for certain directions of the axes of symmetry of a cubically anisotropic body. Thus, for the axes of symmetry of the fourth and second order we have, respectively [8],

$$\cos\alpha_1 = \cos\alpha_2 = 0\,,\quad \cos\alpha_3 = 1\,, \tag{19}$$

$$\cos\alpha_1 = 0\,,\quad \cos\alpha_2 = \cos\alpha_3 = \sqrt{2}/2\,. \tag{20}$$

Substituting Eq. (19) into (18), we obtain

$$v_1 = \sqrt{\left(\frac{A_1}{\rho}\right)},\quad v_2 = v_3 = \sqrt{\left(\frac{A_3}{\rho}\right)},\quad v_4 = \sqrt{\left(\frac{B_1}{j\rho}\right)},\quad v_5 = v_6 = \sqrt{\left(\frac{B_3}{j\rho}\right)}.$$

In the case of Eq. (20) we will have

$$v_1 = \sqrt{\left(\frac{A_3}{\rho}\right)},$$

$$v_{2,3} = \sqrt{\left(\frac{A_3}{\rho}\left(1 + \frac{A_1 - A_3}{2} \pm \sqrt{\left(\frac{(A_1 - A_3)^2}{4} - \frac{A^2}{2} + A\,(A_2 + A_4)\right)}\right)\right)},$$

$$v_4 = \sqrt{\left(\frac{B_3}{j\rho}\right)},$$

$$v_{5,6} = \sqrt{\left(\frac{B_3}{j\rho}\left(1 + \frac{B_1 - B_3}{2} \pm \sqrt{\left(\frac{(B_1 - B_3)^2}{4} - \frac{B^2}{2} + B\,(B_2 + B_4)\right)}\right)\right)}.$$

The velocity of the heat wave v in the micropolar cubically anisotropic medium is equal to $\sqrt{\lambda/c_\varepsilon\tau}$ irrespective of the direction chosen inside of the body.

The method of characteristics used in the present work can be extended to the problems of thermoelasticity with allowance for the connectedness of displacements-microrotations and temperatures.

NOTATION

$\vec{u}$ and $\vec{\varphi}$, displacement and microrotation vectors; ε_{ij}, components of the microdeformation tensor; t_{ij} and m_{ij}, components of the tensors of force and moment stresses; A_k, B_k, $k = 1, 4$, elastic and micropolar constants; β, coefficient that relates mechanical and thermal stresses; ρ, density of the medium; j, moment of inertia; ρl_k, ρf_k, $k = 1, 3$, volumetric moments and mass forces; τ, relaxation time of the heat flux; λ, thermal-conductivity coefficient; c_ε, heat capacity at constant deformation.

REFERENCES

1. V. Novatskii, *Elasticity Theory* [in Russian], Moscow (1975).
2. A. K. Eringen, *Theory of Micropolar Elasticity. Destruction* [in Russian], Vol. 2, Moscow (1975).
3. L. D. Landau and E. M. Lifshits, *Elasticity Theory* [in Russian], Moscow (1987).
4. J. Sharma and H. Singh, *J. Acoust. Am.*, **85**, No. 5, 29-36 (1989).
5. H. H. Sheriet and R. S. Dhailewal, *Q. Appl. Math.*, **38**, No. 1, 192-203 (1990).
6. G. I. Petrashen', *Wave Propagation in Elastic Anisotropic Media* [in Russian], Moscow (1980).
7. V. I. Smirnov, *A Course in Higher Mathematics* [in Russian], Vol. 4, Pt. 2, Moscow (1981).
8. F. I. Fedorov, *Theory of Elastic Waves in Crystals* [in Russian], Moscow (1965).